

UNIVERSIDADE FEDERAL DA INTEGRAÇÃO LATINO-AMERICANA (UNILA) INSTITUTO LATINO-AMERICANO DE ECONOMIA, SOCIEDADE E POLÍTICA (ILAESP)

PROGRAMA DE PÓS-GRADUAÇÃO EM ECONOMIA

COLLUSION SUSTAINABILITY IN COURNOT AND BERTRAND MIXED MODELS

NICOLAU MIGUEL ARMANDO

Foz do Iguaçu
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Orientador: Prof. Dr. Williams Ubaldo Huamani Quispe

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NICOLAU MIGUEL ARMANDO

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Assinatura do Responsável

I dedicate this dissertation to my wife, Elsa J. Amisse, for being my constant partner, for her patience, understanding, and the silent love that sustained me even through the most difficult days of this journey. To my daughters, Sasha and Allana, who taught me the true meaning of perseverance. May this dissertation serve as a testament to how much it is possible to dream and fight for a better future for you and for yourselves. To my family in Mozambique, for being my foundation, even from afar, and for carrying with me the strength of my origins (Ndau). This dissertation is, with all my love and gratitude, for you.

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When the market is shared by a few, the temptation to cooperate often outweighs the duty to compete.
Unknown

RESUMO

Esta dissertação analisa a sustentabilidade do conluio entre firmas privadas em um oligopólio misto com diferenciação horizontal de produtos, diante de uma firma semipública, sob os regimes de Bertrand e Cournot. O objetivo é avaliar como a participação pública e a substituição de produtos afetam o fator de desconto crítico para a manutenção de acordos de colusão. Desenvolve-se um modelo de teoria dos jogos repetidos com estratégia de punição rigorosa, no qual as firmas privadas maximizam lucros e as firmas semipúblicas ponderam lucro e excedente do consumidor. Os resultados revelam que o efeito da participação pública não é uniforme. No modelo de Bertrand, o fator de desconto crítico apresenta relação não linear com a substituição e aumenta com a participação pública, dificultando a colusão. No modelo de Cournot, a relação também é não linear, e o efeito da participação pública varia em função da interação entre o grau de participação pública e o grau de substituição entre os produtos. Abaixo de determinado limiar, maior participação pública pode reduzir o fator de desconto crítico e favorecer a colusão; acima dele, pode aumentá-lo e dificultar sua manutenção. Conclui-se que a sustentabilidade da colusão depende da interação entre estrutura de propriedade, diferenciação e regime de concorrência.

Palavras-chaves: oligopólio misto; sustentabilidade da colusão; concorrência em quantidade; concorrência em preços; diferenciação de produtos.

RESUMEN

Esta tesis analiza la sostenibilidad de la colusión entre empresas privadas en un oligopolio mixto con diferenciación horizontal de productos, frente a una empresa semipública, bajo los regímenes de Bertrand y Cournot. El objetivo es evaluar cómo la participación pública y la sustitución de productos afectan el factor de descuento crítico para el mantenimiento de los acuerdos de colusión. Se desarrolla un modelo de teoría de juegos repetidos con una estrategia de castigo estricto, en el que las empresas privadas maximizan sus beneficios y las empresas semipúblicas ponderan el beneficio y el excedente del consumidor. Los resultados revelan que el efecto de la participación pública no es uniforme. En el modelo de Bertrand, el factor de descuento crítico muestra una relación no lineal con la sustitución y aumenta con la participación pública, lo que dificulta la colusión. En el modelo de Cournot, la relación también es no lineal, y el efecto de la participación pública varía según la interacción entre el grado de participación pública y el grado de sustitución entre productos. Por debajo de un cierto umbral, una mayor participación pública puede reducir el factor de descuento crítico y favorecer la colusión; por encima de este, puede aumentarlo y dificultar su mantenimiento. Se concluye que la sostenibilidad de la colusión depende de la interacción entre la estructura de propiedad, la diferenciación y el régimen competitivo.

Palabras clave: oligopolio mixto; sostenibilidad de la colusión; competencia en cantidad; competencia en precio; diferenciación de producto.

ABSTRACT

This dissertation analyzes the sustainability of collusion between private firms in a mixed oligopoly with horizontal product differentiation, in the face of a semi-public firm, under the Bertrand and Cournot regimes. The objective is to evaluate how public participation and product substitution affect the critical discount factor for maintaining collusion agreements. A repeated games theory model with a strict punishment strategy is developed, in which private firms maximize profits and semi-public firms weigh profit and consumer surplus. The results reveal that the effect of public participation is not uniform. In the Bertrand model, the critical discount factor shows a non-linear relationship with substitution and increases with public participation, hindering collusion. In the Cournot model, the relationship is also non-linear, and the effect of public participation varies depending on the interaction between the degree of public participation and the degree of substitution between products. Below a certain threshold, greater public participation can reduce the critical discount factor and favor collusion; Above that, it can increase and make its maintenance more difficult. It is concluded that the sustainability of collusion depends on the interaction between ownership structure, differentiation, and competitive regime.

Keywords: mixed oligopoly; collusion sustainability; quantity competition; price competition; product differentiation.

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1 INTRODUTION

In a market economy, competition is widely recognized as one of the principal mechanisms to promoting economic efficiency and social welfare. By competing with each other, firms are expected to offer higher-quality products and services at more affordable prices, directly benefiting consumers and fostering innovation. In addition, competitive markets tend to facilitate a more efficient allocation of productive resources, contributing to economic growth and reducing inequalities. However, such benefits are fully realized only in environments where the rivalry between firms is effective and sustainable, a condition that not always observed in practice.

In several strategic sectors, such as mobile telephony, transportation, electricity, and social care services, public and private firms coexist, with distinct institutional objectives. While public firms aim to maximize social welfare, private firms pursue profit maximization exclusively. In this context, as Silva e Pinho (2018) point out, research often focuses on scenarios where private and public firms coexist, differing essentially in their ultimate goal. Starting from this structural premise, the aforementioned authors analyze the conditions that may favor or compromise the sustainability of tacit or explicit collusion between these agents with divergent incentives. This asymmetry influences competitive dynamics and strategic incentives among market participants, directly affecting the likelihood of collusive behavior.

Beyond the distinction based on maximization objectives, the literature on mixed oligopolies frequently highlights a stabilizing or pro-competitive role attributed to public firms. This perception stems from the understanding that, while private firms, drive by profit, have a natural incentive to engage in tacitly collusive coordinated behaviors that are mutually beneficial, public firms, on contrary, are conceptualized as less likely to participate in these schemes. Their orientation towards social welfare (total surplus) acts, in theory, as an intrinsic constraint on agreements that privilege industry income at the expense of consumers. This characteristic is pointed out as one of the virtues of state presence in oligopolistic markets, as it introduces an agent with asymmetrical preferences that can disrupt stable collusive dynamics (SILVA; PINHO, 2018).

The presence of a public firm may have ambiguous effects on collusion:

on the one hand, it reduces the extraordinary profits of private firms, thereby weakening incentives for coordination; on the other, it creates predictability that may support the sustainability of collusive strategies in oligopolistic markets (COLOMBO, 2013).

The practical relevance of this topic is illustrated by cases observed in different international contexts. In the Tunisian mobile telephony market, the privatization of Tunisie Telecom in 2006 transformed a mixed duopoly into a semi-private duopoly and, subsequently, into a market with three private operators, significantly altering incentives for collusion (DEBBICHI; HICHRI, 2024). In the United Kingdom's social care sector, private firms assumed a central role in service provision while coexisting with public operators, thereby influencing market dynamics (CHEN, 2023). In China, privatization reforms initiated in 1978, including in the airport sector, revealed that public firms often operated with bureaucratic inefficiencies and misaligned objectives, affecting the incentives of private firms (CHEN, 2017). Similarly, in Italy, recent antitrust cases showed that private firms managed to establish collusive arrangements even in the presence of public firm, indicating that state involvement does not preclude strategic coordination (COLOMBO, 2016).

These examples demonstrate that the presence of public firms modifies, but does not eliminate, the potential for collusion among private firms, highlighting the need for theoretical and empirical investigation of the factors affecting the sustainability of such agreements.

Collusion among private firms, whether explicit or tacit, predominantly occurs in oligopolistic markets with few firms capable of influencing prices or quantities, thereby undermining allocative efficiency. The industrial organization literature has extensively employed game theory to model such behavior, particularly through repeated games in which the stability of collusive agreements depends on market structure and intertemporal punishment mechanisms, such as "grim trigger" strategies (DENECKERE, 1983; FRIEDMAN, 1971).

Much of the research has focused on Cournot models, based on quantity competition, whereas Bertrand models, in which competition occurs through prices, remain less explored. This distinction is particularly relevant in markets with differentiated products, where the strategic variable directly influences coordination incentives.

Furthermore, many studies consider either static games or games with

infinite horizons, abstracting from the relevance of interactions over finite horizons, which are common in real-world settings such as concession contracts and regulatory reviews.

This dissertation is divided into three chapters. The introduction contextualizes the problem, presents preliminary observations, defines the objectives, and justifies the relevance of the research. The second chapter presents the theoretical and methodological framework, analyzes and discusses the results. Finally, the third chapter presents the concluding remarks.

1.1 **LITERATURE REVIEW**

The market definition proposed by Cabral (1994), which emphasizes the set of firms producing homogeneous or closely related goods, provides an appropriate starting point for the analysis of strategic interaction in oligopolistic environments. From this perspective, the market is not merely a physical or geographical space, but rather a network of strategic relationships among firms whose decisions jointly determine prices, quantities, and the allocation of economic surplus. This approach is particularly relevant for oligopolistic markets, where firm´s choices are inherently interdependent and cannot be made in isolation.

In a similar vein, Kupfer e Hasenclever (2002) conceive the market as an abstract institutional space in which prices and quantities emerge from the interaction between supply and demand, shaping profits, resource allocation, and social welfare. When market concentration is high, the structural characteristics of firms become inseparable from the strategic logic governing their behavior. This insight is especially important in markets where the state acts not only as a regulator but also as a market participant, as is the case in mixed oligopolies.

The distinction between public and private firms plays a central role in this context. According to Wamsley e Zald (1973), these firms differ fundamentally in terms of ownership, financing, and, above all, institutional objectives. While private firms are typically modeled as profit maximizer, public firms are often assumed to internalize social objectives, such as consumer surplus or overall welfare. These differences in objectives translate into distinct strategic incentives, affecting firms competitive conduct and their responses to rival´s actions. As argued by Pinho (2018), private firms tend

to display a higher propensity toward coordinated behavior, whereas public firms are generally less inclined to engage in collusive practices, partly due to state oversight and their broader objectives.

Understanding strategic interaction in such environments requires an explicit game-theoretic framework. As emphasized by Cabral (1994) , the formal analysis of firms behavior begins with the definition of a game composed of players, strategy sets, payoff functions, and rules of interaction. Each firm chooses its strategy independently, taking into account the anticipate reactions of its rivals. The outcomes of this strategic process is the Nash equilibrium (NASH, 1951), in which no player can unilaterally improve its payoff. In oligopolistic markets, the Nash equilibrium provides a natural benchmark for analyzing competitive outcomes.

However, the Nash equilibrium does not exhaust the set of relevant outcomes when firms interact repeatedly over time. In many oligopolistic industries, competition is not a one-shot interaction but rather a long-term process, which opens the possibility for implicit coordination among firms. In such settings, the sustainability of collusion becomes a central analytical issue, as repeated interaction may allow firms to support cooperative outcomes through credible punishment strategies, despite short-run incentives to deviate.

The analysis of collusion in markets with public and private firms builds upon earlier contributions to the theory of mixed oligopolies. Merrill e Schneider (1966) were among the first to examine the interaction between public and private firms in concentrated markets, distinguishing between fully private, fully state-owned, and intermediate ownership structures. Their analysis showed that state entry through the acquisition of an incumbent firm could discipline the market by increasing output and reducing prices, thereby laying the groundwork for subsequent debates on the role of public firms as instruments of economic policy.

Subsequent studies expanded this line of inquiry by incorporating strategic behavior more explicitly. Dixit (1979) introduced a framework with differentiated products and strategic leadership, showing how incumbents could deter entry through their output choices. Building on this structure, Singh e Vives (1984) compared competition in quantities and prices, demonstrating that, for substitute goods, Bertrand competition tends to generate higher welfare, whereas Cournot competition may be

more efficient when goods are complements. Later contributions, such as Häckner et al. (1999), showed that this dichotomy is sensitive to assumptions regarding the number of firms and product characteristics, indicating that the relative aggressiveness of price and quantity competition is not unambiguous.

Within the mixed oligopoly literature, Fraja e Delbono (1989) analyzed markets in which private firms maximize profits while public firms pursue social welfare, concluding that nationalization may be welfare-enhancing in highly concentrated markets, whereas profit maximization by the public firm may be preferable in less concentrated environments. Extending this framework, Matsumura (1998) introduced the concept of partial privatization, modeling public firms as maximizing a weighted average of profit and social welfare. His results show that neither full nationalization nor full privatization is generally optimal, a finding that has become a cornerstone of the modern mixed oligopoly literature.

More recent contributions have shifted attention toward the sustainability of collusion in mixed markets. Colombo (2016) analyzes a dynamic model in which private firms attempt to sustain collusion in the presence of a public firm that does not participate in the cartel. The author shows that an increase in state ownership may paradoxically strengthen collusion among private firms by making post-deviation punishments more severe, thereby reducing incentives to deviate. In contrast, Silva e Pinho (2018) demonstrate that when public firms internalize social welfare, their expansion of output may offset the effects of collusive output restriction by private firms, rendering collusion less profitable or even unsustainable. These contrasting results highlight that the impact of public firms on collusion critically depends on the competitive regime, cost structure, and the objectives embedded in the public firms behavior.

Evidence supporting this parameter-dependent view is provided by Debbichi e Hichri (2024), who analyze a mixed duopoly in the Tunisian mobile telecommunications market. Their model introduces a critical threshold of preference for collusion that varies with cost structure and interconnection tariffs. The authors show that collusion is more difficult to sustain in private duopolies than in mixed ones, particularly when costs are linear rather than quadratic, reinforcing the idea that ownership structure and cost convexity jointly shape collusive incentives. Importantly, their empirical application highlights the relevance of mixed oligopoly models for developing

economies, where state participation remains substantial and regulatory capacity is limited.

A complementary strand of the literature examines the role of product differentiation in cartel stability. Symeonidis (2002) shows that greater horizontal differentiation and multi-product competition tend to increase the critical discount factor required to sustain collusion, making cooperative outcomes more difficult to maintain under both price and quantity competition. Although this analysis does not explicitly focus on mixed oligopolies important insights into how product heterogeneity affects strategic incentives to collude.

The role of market structure in cartel stability has also been examined through spatial competition models. Andree et al. (2018) extend the analysis of cartel sustainability by introducing convex production costs into a spatial price discrimination framework. Their results indicate that greater cost convexity enhances cartel stability, even when firms are allowed to relocate over time, although endogenous relocation tends to weaken collusive outcomes. These findings complement the insights of Symeonidis (2002) by showing that structural features beyond product differentiation, such as spatial dispersion and cost curvature, play a decisive role in shaping the incentives to collude.

A particularly challenging implication for competition policy emerges from the literature on public-private collusion. Mota, Silva e Pinho (2019), Mota, Silva e Pinho (2023) show that, under certain conditions, collusion between a public and a private firm may increase total surplus and even consumer surplus by correcting productive inefficiencies arising from the public firms expansionary behavior under non-cooperation. These results questions the conventional presumption that collusion is necessarily welfare-reducing and suggest that antitrust authorities may need to account for institutional objectives when assessing collusive conduct in mixed markets.

Despite these advances, the existing literature remains fragmented. Most studies on collusion in mixed oligopolies focus on quantity competition and abstract from a systematic comparison with price competition. Moreover, the interaction between public ownership, product differentiation, and the sustainability of collusion indifferent competitive regimes has received limited attention. This gap motivates the present dissertation, which aims to evaluate how the presence of a public firm affects the sustainability of collusion between private firms in mixed oligopoly models under Cournot

and Bertrand competition with horizontal product differentiation, and to assess the implications of these results for antitrust policy and economic regulation.

1.2 **PRELIMINARY REMARKS**

This dissertation aims to analyze the sustainability of collusion in mixed oligopoly models from a theoretical perspective. The choice of topic is justified by the need to understand how the simultaneous presence of public and private firms influences competitive dynamics and the stability of collusive agreements in mixed oligopolies.

Although the literature recognizes the role of public firms in promoting social welfare Matsumura (1998), Fraja e Delbono (1989), their effects on the viability of collusion remain insufficiently explored and continue to spark recent debates (COLOMBO, 2016).

The proposed research is based on microeconomic modeling, seeking to identify the conditions under which collusion can be sustained in different competitive contexts. In this sense, the work draws on the theoretical tradition that analyzes the strategic interaction between firms in mixed markets. By adopting this approach, we aim to offer relevant conceptual contributions to the academic debate and to the formulation of competition and regulation policies, especially in scenarios where public and private firms coexist.

1.3 **THE PROBLEM AND ITS IMPORTANCE**

The literature on industrial organization has widely used game theory to model collusive behavior, especially through repeated games. Since the classic works of Friedman (1971) and Deneckere (1983), it has been recognized that the stability of collusive agreements depends both on market structure and on intertemporal punishment mechanisms, often implemented by grim trigger strategies.

Despite theoretical advances, most analyses focus on markets composed exclusively of private firms, leaving the role of public companies in competitive dynamics in the background. The literature still lacks consensus on the impact of a state-owned firm oriented toward social welfare the strategic incentives for cooperation

between private firms. In this context, the central question of this research arises: does the presence of a public firm facilitate or hinder the sustainability of collusion between private firms in oligopolistic markets with product differentiation?

The relevance of this study is both theoretical and practical. From a theoretical point of view, investigating the effect of public firms on the sustainability of collusion contributes to the deepening of the literature on Industrial Organization, providing insights into mixed markets and the strategic interactions between private and state agents.

From a practical perspective, understanding these effects has direct implications for the formulation of economic and antitrust regulation policies, considering that the actions of public companies can play a crucial role in promoting more competitive markets and social welfare. Thus, the study expands knowledge about intertemporal incentives for collusion, while providing a basis for economic policy decisions in contexts where private and public actors coexist.

1.4 **OBJECTIVES**

1.4.1 *General Objective*

The general objective of this study is to evaluate the impact of the presence of a public firm on the sustainability of collusion between private firms, for which we will use a mixed oligopoly model with horizontal differentiation of the Cournot and Bertrand types. We seek to analyze how the coexistence of public firms influences or dis-incentivizes private firms from entering into and maintaining collusive agreements.

1.4.2 *Specific Objectives*

- To compare the Cournot and Bertrand competition regimes, examining how different market structures influence the stability of collusive agreements, considering that, in the Cournot regime, competition occurs in quantities, while in the Bertrand regime it occurs in prices.
- To analyze the influence of product differentiation on the viability of collusion, recognizing that different levels of product heterogeneity can alter the strategic

incentives of private and public companies and, consequently, the probability of maintaining collusive agreements.

- Assesses the implications of these analyses for the formulation of antitrust and economic regulation policies, in order to provide theoretical insights into how the coexistence of public and private firms can affect the effectiveness of regulatory instruments and interventions in mixed markets, considering the strategic conditions that favor or hinder collusion.

2 METHODOLOGY

This dissertation adopts a theoretical-analytical methodology based on oligopoly models. The method consists of constructing a formal industrial organization model that includes three firms in an oligopolistic market, which allows for the analysis of the sustainability of collusive agreements under different institutional configurations (private, public, or semi-public firms).

The proposed model is based on the theoretical framework of Colombo (2013), incorporating elements of the Bertrand model with product differentiation and combining characteristics of the Cournot and Bertrand models. In this way, the model captures both strategic quantity interactions and the effects of price competition when products have different degrees of sustainability.

The model is structured around inverse demand functions, firm profit functions, and an aggregate surplus function. Based on this formalization, maximization techniques are applied to derive the Nash-Cournot and Nash-Bertrand equilibrium conditions. This allows for an investigation into the effects of the degree of sustainability between goods, strategic interaction, and the presence of public firms in the market.

Furthermore, a comparative static analysis is conducted to examine how changes in the model's parameters, especially the degree of sustainability, affect the sustainability of collusion and social welfare outcomes.

The interpretation of the results is based on classical and contemporary literature on industrial organization regarding the theory of economic collusion, using the sub-game perfect equilibrium in a grim trigger strategy.

2.1 MODEL

Consider a market populated by a representative consumer whose preferences are described by the following utility function, adapting the frameworks of Dixit (1979) and Häckner et al. (1999):

$$U(q_1, q_2, q_3) = a \left[\sum_{i=1}^{3} q_i \right] - \frac{1}{2} \left[\sum_{i=1}^{3} q_i^2 + 2\theta \sum_{i=1}^{3} \sum_{j \neq i}^{3} q_i q_j \right] + I. \tag{2.1}$$

where q_i denotes the quantity of good $i \in \{1, 2, 3\}$, and $a > 0$ represents the intrinsic market size or demand intercept. The parameter $\theta \in (0, 1)$ captures the degree of

substitution between products i and j ($i, j \in \{1, 2, 3\}, i \neq j$).

The consumer's optimization problem is constrained by the budget allocation defined as $p_1q_1 + p_2q_2 + p_3q_3 + I = M$, where p_i represents the price of good i, M denotes total consumer income, and the price of the numéraire good, I, is normalized to 1. By maximizing this utility function subject to the budget constraint, we derive the following inverse demand system:

$$p_i = a - \left[q_i + \theta \sum_{j \neq i} q_j \right], \quad i = 1, 2, 3 \tag{2.2}$$

Consider the boundary cases of $\theta = 0$ and $\theta = 1$. When $\theta = 0$, the inverse demand function simplifies to $p_i = a - q_i$, implying that the demand for good i is completely independent of the quantities produced by its rivals. In this edge case, the market structure effectively dissolves into three isolated monopolies, representing independent product markets. Conversely, when $\theta = 1$, the inverse demand becomes $p_i = a - (q_i + \sum_{j \neq i} q_j)$, meaning that the goods are perfect substitutes. Under this parameter configuration, consumers perceive no qualitative differentiation between the goods, collapsing the industry into a standard market for homogeneous products.

Subsequently, by inverting the inverse demand system, we obtain the direct demand system:

$$q_i = \frac{(1-\theta)a - (1+\theta)p_i + \theta \sum_{j \neq i} p_j}{(1-\theta)(1+2\theta)}, \quad i = 1, 2, 3 \tag{2.3}$$

Crucially, the inverse and direct demand systems characterize the strategic environments for Cournot and Bertrand competition, respectively. Finally, substituting the direct demand functions back into the consumer's objective utility, we obtain the consumer surplus (SC). Following Colombo (2013), this welfare metric is explicitly formulated as:

$$SC = \sum_{i=1}^{3} aq_i - \frac{1}{2} \sum_{i=1}^{3} q_i^2 - \theta \sum_{i \neq j} q_i q_j \tag{2.4}$$

Equation (2.4) shows that consumer welfare depends directly on the total quantities consumed and the degree of product substitution. As products become closer substitutes (θ increases), the third which is negative, increases, reflecting how a loss of products reduces consumer's total satisfaction.

2.1.1 Agent interaction

Consider an oligopolistic sector with horizontally differentiated products, served by three firms, $i \in \{1, 2, 3\}$. Firms 1 and 2 are private, profit-maximizing entities, whereas firm 3 operates as a partially privatized (or semi-public) enterprise.

The strategic interaction between these agents occurs within either a Cournot or a Bertrand framework. Let the strategic variable of firm i be denoted by s_i. In subsequent sections, this strategy variable is explicitly defined based on the competitive regime: under Cournot competition, firms compete by setting quantities ($s_i = q_i$), whereas under Bertrand competition, they compete by setting prices ($s_i = p_i$).

Under this asymmetric institutional structure, the two private firms ($i = 1, 2$) seek strictly to maximize their respective economic profits:

$$\pi_i(s_i, s_{-i}) = (p_i - c_i)q_i, \quad i = 1, 2 \tag{2.5}$$

where $c_i = c = 0$ denotes the constant marginal cost of production, assumed to be symmetric across all market participants. In contrast, the semi-public firm 3 aims to maximize a convex combination of its own producer surplus (π_3) and the consumer surplus (SC). This specific objective function reflects its hybrid mandate, balancing commercial viability with broader social goals:

$$\Gamma_3(s_3, s_{-3}) = \alpha SC(s_3, s_{-3}) + (1 - \alpha)\pi_3(s_3, s_{-3}) \tag{2.6}$$

where the parameter $\alpha \in [0, 1]$ represents the degree of private ownership. Consequently, the firm operates as a purely public entity maximizing social objectives when $\alpha = 1$, and converges to a standard private profit-maximizing firm when $\alpha = 0$.

While the choice of strategic variables shapes the equilibrium outcomes, the underlying incentives to compete, cooperate, or unilaterally cheat are equally critical. These strategic tensions dictate the intertemporal stability of the market, establishing the economic conditions under which a collusive agreement among the private entities can remain sustainable within a mixed oligopoly setting.

2.1.2 Collusion sustainability

To analyze the stability of the cartel, we consider an infinitely repeated game framework where firms evaluate their streams of payoffs using a common discount factor $\delta \in (0,1)$. Within this setting, the two symmetric private firms, $i \in \{1,2\}$, coordinate on a collusive strategy (s_i^C), while the semi-public firm 3 remains outside the agreement and competes in every period.

The intertemporal behavior of the private firms is governed by a standard grim trigger strategy, following the classic supergame framework established by Friedman (1971). Both firms maintain cooperation (s_i^C) as long as no deviation is observed; however, if either firm unilaterally cheats, the agreement permanently collapses, and both private entities revert to the stage-game Nash equilibrium strategy (s_i^N) forever.

Following Huamani, Braga e Pinha (2024), a collusive agreement among the private firms is said to be sustainable if, for every private firm $i \in \{1,2\}$, there exists a discount factor $\delta \in (0,1)$ such that $\delta \geq \delta^*$, where $\delta^* = \max\{\delta_1, \delta_2\}$, which satisfies the following incentive compatibility constraint (ICC)[1]:

$$\frac{\pi_i^C}{1-\delta} \geq \pi_i^D + \frac{\delta \pi_i^N}{1-\delta} \iff \delta \geq \delta_i^* = \frac{\pi_i^D - \pi_i^C}{\pi_i^D - \pi_i^N} \tag{2.7}$$

where π_i^D, π_i^C, and π_i^N represent the deviation, collusive, and Nash competition profits, respectively[2]. Given that the private firms are perfectly symmetric in costs and demands, they face an identical incentive structure ($\delta_1 = \delta_2$), meaning that the stability of the entire cartel is governed by a single, unique critical discount factor threshold, $\delta^* = \delta_1^* = \delta_2^*$.

2.1.2.1 *Bertrand Competition*

In this section, we analyze the strategic interactions when prices are the strategic variables, setting $s_i = p_i$ and $s_{-i} = p_{-i}$ to map our framework directly to the pricing regime. Under this competitive setup, the firms' market demand is characterized by the direct demand system derived in (2.3).

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[1] The underlying intertemporal condition for each private firm i is defined as $V_i^C \geq V_i^D \iff \sum_{t=0}^{\infty} \delta^t \pi_i^C \geq \pi_i^D + \sum_{t=1}^{\infty} \delta^t \pi_i^N$.

[2] To guarantee that the collusion dilemma is economically meaningful, the payoffs are assumed to satisfy the strict hierarchical order: $\pi_i^D > \pi_i^C > \pi_i^N$.

Lema 2.1 *The Bertrand-Nash equilibrium is:*

$$p_j^N = \frac{(1-\theta)\left((2+3\theta)-(1+2\theta)\,\alpha\right)a}{2\,(2+3\theta)-(2+3\theta+\theta^2)\alpha}, \quad p_3^N = \frac{(1-\theta)\left((2+3\theta)-(2+\theta)\,\alpha\right)a}{2\,(2+3\theta)-(2+3\theta+\theta^2)\alpha}, \quad j=1,2 \tag{2.8}$$

Then, the competitive profits are:

$$\pi_j^N = \frac{(1-\theta^2)\left((2+3\theta)-(1+2\theta)\,\alpha\right)^2 a^2}{(1+2\theta)\left(2\,(2+3\theta)-(2+3\theta+\theta^2)\alpha\right)^2}, \quad j=1,2$$

Proof 2.1 *The Bertrand-Nash equilibrium is characterized by solving the system of best-response functions in prices. For the private firms $i,j \in \{1,2\}$ (with $i \neq j$) and the semi-public firm 3, the reaction functions are given by:*

$$p_i = \frac{(1-\theta)a+\theta(p_j+p_3)}{2(1+\theta)}, \quad p_3 = \frac{(1-\theta)(1-\alpha)a+\theta(p_i+p_j)}{(1+\theta)(2-\alpha)} \tag{2.9}$$

Solving this system (2.9)*, the equilibrium prices p_j^N and p_3^N stated in Lemma 1. Substituting these back into the profit functions concludes the proof.*

Having established the non-cooperative benchmark, we now characterize the cooperative regime, where the two private firms coordinate their pricing decisions to maximize joint profits, anticipating the non-cooperative reaction of the semi-public competitor.

Lema 2.2 *The cooperative equilibrium is:*

$$\begin{aligned} p_j^C &= \frac{(1-\theta)\Big((2+3\theta)-(1+2\theta)\alpha\Big)a}{2\Big(2(1+\theta-\theta^2)-(1+\theta)\alpha\Big)}, \qquad j=1,2, \\ p_3^C &= \frac{(1-\theta)\Big((2+4\theta+\theta^2)-(3+4\theta)\alpha+(1+\theta)\alpha^2\Big)a}{(2-\alpha)(1+\theta)\Big(2(1+\theta-\theta^2)-(1+\theta)\alpha\Big)} \end{aligned} \tag{2.10}$$

Then, the cooperative profits are:

$$\pi_j^C = \frac{(1-\theta)\left((2+3\theta)-(1+2\theta)\,\alpha\right)^2 a^2}{4\,(2-\alpha)\,(1+\theta)\,(1+2\theta)\,(2\,(1+\theta-\theta^2)-(1+\theta)\,\alpha)}, \quad j=1,2$$

Proof 2.2 *To find the cooperative equilibrium, we solve the reaction system derived from the first-order conditions of joint profit maximization and the semi-public firm's objective function:*

$$\frac{(1-\theta)((2+3\theta)-(1+2\theta)\alpha)a-\Big[2(2+4\theta+\theta^2)-2(1+\theta)^2\alpha\Big]p_i+[2\theta((2+3\theta)-(1+\theta)\alpha)]p_j}{(1+2\theta)(1-\theta^2)(2-\alpha)} = 0 \tag{2.11}$$

Imposing symmetry ($p_i^C = p_j^C$) and solving this system yields the equilibrium prices p_j^C and p_3^C in Lemma 2.2. Substituting these into the profit functions completes the proof.
(2.11)

Although the cooperative agreement yields higher joint payoffs, it is inherently vulnerable to cheating. To assess the stability of this collusion, we must determine the maximum short-run payoff a private firm can obtain by unilaterally deviating from the agreement while its rival remains cooperative.

Lema 2.3 *The deviate equilibrium is:*

$$p_i^D = \begin{cases} \frac{(1-\theta)((2+3\theta)-(1+2\theta)\alpha)\left(2(2+3\theta)-\left(2+3\theta+\theta^2\right)\alpha\right)a}{4(2-\alpha)(1+\theta)^2(2(1+\theta-\theta^2)-(1+\theta)\alpha)}, & \text{if } 0<\theta<\bar{\theta}(\alpha) \\ \frac{(1-\theta)\left(\left(1-\theta^2\right)\alpha-2\left(1-2\theta^2\right)\right)((2+3\theta)-(1+2\theta)\alpha)a}{2\theta(1+\theta)(2(1+\theta-\theta^2)-(1+\theta)\alpha)(2-\alpha)}, & \text{if } \bar{\theta}(\alpha)\leq\theta<1 \end{cases} \tag{2.12}$$

Then, the deviate and damage profits are:

$$\pi_i^D = \begin{cases} \frac{(1-\theta)((2+3\theta)-(1+2\theta)\alpha)^2\left(2(2+3\theta)-\left(2+3\theta+\theta^2\right)\alpha\right)^2a^2}{16(2-\alpha)^2(1+\theta)^3(1+2\theta)(2(1+\theta-\theta^2)-(1+\theta)\alpha)^2}, & \text{if } 0<\theta<\bar{\theta}(\alpha) \\ \frac{(1-\theta)\left(\left(1-\theta^2\right)\alpha-2\left(1-2\theta^2\right)\right)((2+3\theta)-(1+2\theta)\alpha)^2a^2}{4\theta^2(1+\theta)^2(2(1+\theta-\theta^2)-(1+\theta)\alpha)(2-\alpha)^2}, & \text{if } \bar{\theta}(\alpha)\leq\theta<1 \end{cases}$$

$$\pi^{Di} = \begin{cases} \frac{(1-\theta)((2+3\theta)-(1+2\theta)\alpha)^2\left(2\left(2+4\theta-\theta^2-4\theta^3\right)-(1+\theta)\left(2+2\theta-\theta^2\right)\alpha\right)a^2}{8(2-\alpha)(1+\theta)^2(1+2\theta)(2(1+\theta-\theta^2)-(1+\theta)\alpha)^2}, & \text{if } 0<\theta<\bar{\theta}(\alpha) \\ 0, & \text{if } \bar{\theta}(\alpha)\leq\theta<1 \end{cases}$$

Proof 2.3 *To find the optimal deviation, we assume firm j charges its collusive price p_j^C while firm i sets p_i^D. For $\theta \in (0, \bar{\theta}(\alpha))$, the unconstrained deviation price p_i^D is derived from the first-order condition, yielding:*

$$\frac{((1-\theta)((2+3\theta)-(1+2\theta)\alpha)(2(2+3\theta)-(1+\theta)(2+\theta)\alpha))a-4\left(2\left(1+\theta-\theta^2\right)-(1+\theta)\alpha\right)(1+\theta)^2(2-\alpha)p_i}{2(1+2\theta)(1-\theta^2)(2(1+\theta-\theta^2)-(1+\theta)\alpha)(2-\alpha)} = 0 \tag{2.13}$$

For $\theta \in [\bar{\theta}(\alpha), 1)$, the unconstrained price would yield negative demand for the cheated firm j. Under this regime, the optimal deviation price is constrained by setting firm j residual demand to zero, $q_i(p_i, p_j^C, p_3^C) = 0$, which implies:

$$\frac{(1-\theta)\alpha-(1+\theta)p_j^C+\theta(p_i+p_3^C)}{(1-\theta)(1+2\theta)} = 0$$

Solving these respective equations for each parametric region yields the deviation prices and profits presented in Lemma 2.3.

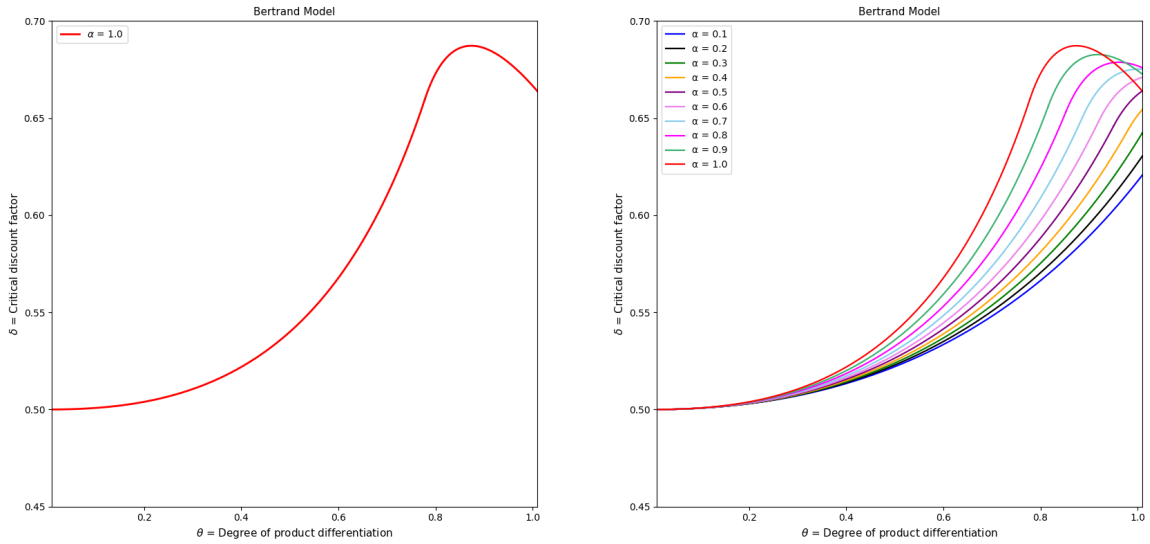


Figura 1 – The effect of the degree of public ownership on collusion sustainability at different levels of product differentiation – Bertrand model.

By characterizing the pricing decisions and payoffs across the competitive, cooperative, and deviation stages (derived in Lemmas 2.1, 2.2 and 2.3, respectively), we have established the necessary analytical components to evaluate the stability of the collusion between firms 1 and 2. Substituting these profits into the incentive compatibility constraint (2.7) yields the critical discount factor under price competition:

$$\delta_p(\theta) \geq \begin{cases} \frac{\left(2(2+3\theta)-\alpha\left(2+3\theta+\theta^2\right)\right)^2}{4(8+24\theta+17\theta^2-4\theta^3-4\theta^4)-4(1+\theta)(8+16\theta+5\theta^2-2\theta^3)\alpha+(8+8\theta+\theta^2)(1+\theta)^2\alpha^2}, & \text{if } \ 0<\theta<\bar{\theta}(\alpha) \\ \frac{\left(2\left(1+2\theta-\theta^2-3\theta^3\right)-(1+\theta)\left(1+\theta-\theta^2\right)\alpha\right)(2(2+3\theta)-(1+\theta)(2+\theta)\alpha)^2}{I_1-I_2\alpha+I_3\alpha^2-I_4\alpha^3}, & \text{if } \ \bar{\theta}(\alpha)\leq\theta<1 \end{cases} \tag{2.14}$$

where $I_1 = 8\left(4+20\theta+29\theta^2-6\theta^3-46\theta^4-32\theta^5-8\theta^6-4\theta^7\right)$, $I_2 = 4(12+48\theta+47\theta^2 -31\theta^3-66\theta^4-28\theta^5-8\theta^6)(1+\theta)$, $I_3 = 2\left(12+36\theta+19\theta^2-24\theta^3-22\theta^4-8\theta^5\right)(1+\theta)^2$ and $I_4 = \left(4+8\theta+\theta^2-3\theta^3-2\theta^4\right)(1+\theta)^3$. Since private firms are symmetric, they are governed by a single ICC. Furthermore, collusion sustainability depends on both the participation of the public firm and the degree of product differentiation.

Figure 1 illustrates the relationship between the degree of substitutability of products (θ) and the critical discount factor (δ^*) under Bertrand price competition, considering different degrees of participation of the public firm (α). In the Bertrand competition structure, the trajectory of δ^* exhibits a non-monotonic dependence on the substitutability of goods. When products are strongly differentiated ($\theta \to 0$), the choice of the control variable reduces the strategic interdependence between private firms and the public firm, converging the collusion sustainability threshold to $\delta^* = 0.5$. As

the substitutability of goods increases ($\theta > 0$), there is a sharp increase in the critical discount factor to maintain the tacit agreement between private firms. This rise reaches a peak at high levels of homogeneity (around $\theta \approx 0.85$ for $\alpha = 1.0$) before showing a downward inflection in the final stages of perfect substitution ($\theta \to 1$).

The increase in the degree of public ownership (α) monotonically shifts the critical discount factor curve upward over most of the θ domain. For higher levels of α, the public firm acts aggressively in pricing to maximize social welfare, setting prices close to marginal cost. This state pricing generates two opposing forces on the stability of the collusion: i gains from deviation (π^D): the presence of a public firm restricts the profit margin that private firms can obtain during the period in which they comply with the collusive agreement (π^C). However, the incentive to unilaterally deviate (π^D) becomes relatively attractive, since cutting prices slightly below the collusive level allows capturing a larger range of demand under moderate and high differentiation; and (ii) the severity of the punishment (π^N): the punishment phase, governed by the non-cooperative Bertrand-Nash equilibrium in repeated games with the grim trigger strategy Friedman (1971), is severe. As the public firm pursues social welfare, the punishment equilibrium forces long-term profits to compressed levels.

As demonstrated by Deneckere (1983) and expanded upon by Singh e Vives (1984), under price competition, the immediate gain from a unilateral price cut exceeds the loss brought about by future punishment when goods become close substitutes. In the context of mixed oligopoly Matsumura (1998), Colombo (2013), the socially oriented actions of the public firm amplify this incentive asymmetry as α grows; the reduction in collusive surplus income surpasses the threat of punishment, making the maintenance of collusion tacitly more difficult, which requires a lower degree of impatience (δ^* higher) on the part of private firms.

The inflection point observed when θ approaches unity ($\theta \to 1$) results from saturation in the gain from deviation. When goods are nearly perfect substitutes, the marginal pricing imposed by the presence of the public firm reduces the viable collusive margin to almost zero, substantially equating diversion profits with Nash equilibrium profits, marginally reducing the critical discount factor threshold.

2.1.2.2 *Cournot Competition*

In this section, we analyze the strategic interactions when quantities are the strategic variables for all firms in the industry, setting $s_i = q_i$ for $i \in \{1, 2, 3\}$. Under this competitive environment, the market is characterized by the inverse demand system derived in (2.2).

The following three lemmas (Lemmas 2.4, 2.5 and 2.6) corroborate the equilibrium quantities and payoffs under quantity competition, aligning with the foundational framework established by Colombo (2013).

Lema 2.4 *The Nash equilibrium is:*

$$q_j^N = \frac{(2-\theta-\alpha)\,a}{2\,(2+\theta-\theta^2)-(2+\theta)\,\alpha}, \quad q_3^N = \frac{(2-\theta)\,a}{2\,(2+\theta-\theta^2)-(2+\theta)\,\alpha}, \quad j=1,2 \tag{2.15}$$

Then, the competitive profits are:

$$\pi_j^N = \frac{(2-\theta-\alpha)^2\,a^2}{(2\,(2+\theta-\theta^2)-(2+\theta)\,\alpha)^2}, \quad j=1,2$$

Proof 2.4 *To find the Cournot-Nash equilibrium, we solve the system of reaction functions for the private firms $i, j \in \{1, 2\}$ (with $i \neq j$) and the semi-public firm 3:*

$$q_i = \frac{a-\theta(q_j+q_3)}{2}, \quad i,j \in 1,2, i \neq j \quad q_3 = \frac{a-\theta(q_1+q_2)}{2-\alpha} \tag{2.16}$$

Solving this system yields the equilibrium quantities in Lemma 2.4. Substituting these back into the private firms' profits completes the proof.

Once the competitive benchmark has been established, we characterize the cooperative scenario where both private firms coordinate their outputs to maximize joint profits.

Lema 2.5 *The cooperative equilibrium is:*

$$q_j^C = \frac{(2-\theta-\alpha)\,a}{2\,(2\,(1+\theta-\theta^2)-(1+\theta)\,\alpha)}, \quad q_3^C = \frac{(2-\theta^2-\alpha)\,a}{(2-\alpha)\,(2\,(1+\theta-\theta^2)-(1+\theta)\,\alpha)}, \quad j=1,2 \tag{2.17}$$

Then, the cooperative profits are:

$$\pi_j^C = \frac{(2-\theta-\alpha)^2\,a^2}{4\,(2-\alpha)\,(2\,(1+\theta-\theta^2)-(1+\theta)\,\alpha)}, \quad j=1,2$$

Proof 2.5 *To find the cooperative equilibrium, the private firms maximize joint profits, while the semi-public firm optimizes its objective function. For $i, j \in \{1, 2\}$ with $i \neq j$, the resulting reaction functions are:*

$$\frac{(2-\theta-\alpha)a - 2(2-\alpha-\theta^2)q_i - 2\theta(2-\theta-\alpha)q_j}{2-\alpha} = 0, \quad q_3 = \frac{a-\theta(q_i+q_j)}{2-\alpha} \tag{2.18}$$

Imposing symmetry ($q_i^C = q_j^C$) and solving this system yields the cooperative quantities in Lemma 2.5. Substituting these back into the private profits completes the proof.

Although the cooperative agreement yields higher joint profits, each private firm has an individual incentive to unilaterally cheat and maximize its short-run payoff.

Lema 2.6 *The deviate equilibrium is:*

$$q_j^D = \frac{(2-\theta-\alpha)\left(2(2+\theta-\theta^2)-(2+\theta)\alpha\right)a}{4(2-\alpha)\left(2(1+\theta-\theta^2)-(1+\theta)\alpha\right)}, \quad j, -j = 1, 2, \;\; j \neq -j \tag{2.19}$$

Then, the deviate and damage profits are:

$$\pi_j^D = \frac{(2-\theta-\alpha)^2\left(2(2+\theta-\theta^2)-(2+\theta)\alpha\right)^2 a^2}{16(2-\alpha)^2\left(2(1+\theta-\theta^2)-(1+\theta)\alpha\right)^2}$$

$$\pi_{-jD} = \frac{(2-\theta-\alpha)^2\left(2(2+2\theta-3\theta^2+\theta^3)-(2+2\theta-\theta^2)\alpha\right)a^2}{8(2-\alpha)\left(2(1+\theta-\theta^2)-(1+\theta)\alpha\right)^2}$$

Proof 2.6 *To determine the optimal deviation, we assume that private firm j produces its cooperative output q_j^C, while private firm i unilaterally cheats by setting q_i^D. For $i, j \in \{1, 2\}$ with $i \neq j$, the first-order condition for the deviating firm yields:*

$$\frac{(2-\theta-\alpha)(2(2-\theta)(1+\theta)-(2+\theta)\alpha)a - 4(2-\alpha)(2(1+\theta(1-\theta))-(1+\theta)\alpha)q_i}{2(2-\alpha)(2(1+\theta-\theta^2)-(1+\theta)\alpha)} = 0 \tag{2.20}$$

Solving this equation for q_i yields the deviation quantity in Lemma 2.6. Substituting these quantities back into the profit functions completes the proof.

By utilizing the equilibrium profits derived in Lemmas 2.4, 2.5 and 2.6 , we can formulate the incentive compatibility constraint that guarantees collusion sustainability for private

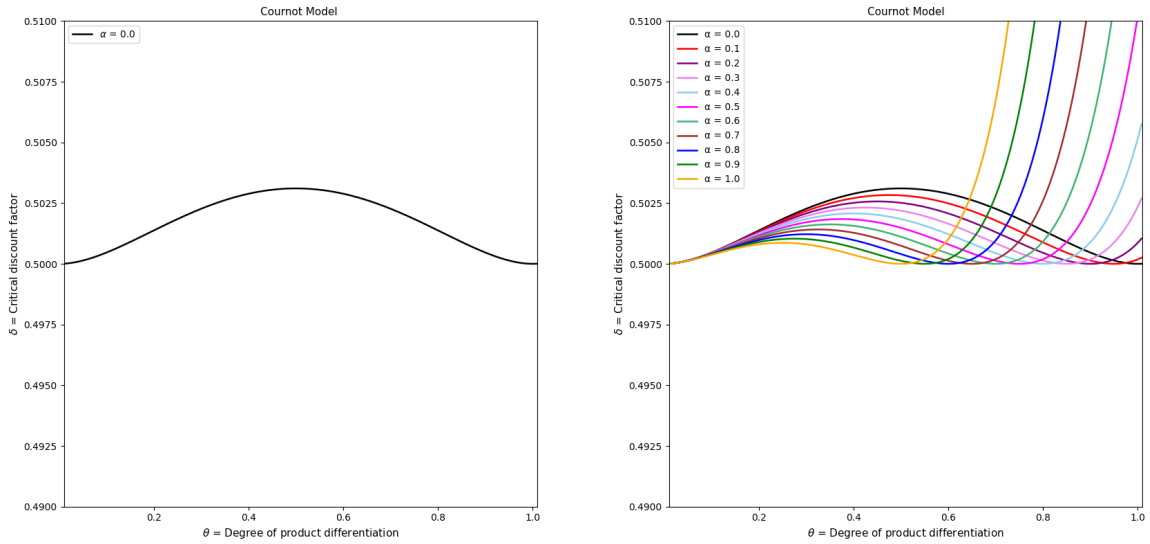


Figura 2 – The effect of the degree of public ownership on collusion sustainability at different levels of product differentiation – Cournot model.

firms 1 and 2, which aligns with the critical discount factor threshold established by Colombo (2013).

$$\delta_q(\alpha,\theta) \geq \frac{\left(2\left(2-\theta\right)\left(1+\theta\right)-\left(2+\theta\right)\alpha\right)^2}{4\left(8+8\theta-7\theta^2-2\theta^3+\theta^4\right)-\left(32+32\theta-12\theta^2-4\theta^3\right)\alpha+\left(8+8\theta+\theta^2\right)\alpha^2}$$

This incentive compatibility constraint reveals that the threshold for collusion sustainability is determined by both the level of state involvement (α) and the degree of product differentiation (θ). In his seminal analysis, Colombo (2016) demonstrated that when the third firm operates with certain degree of public interest, the relationship between the product substitutability and the feasibility of collusion is non-monotonic. Specifically, the presence of a socially oriented competitor reshapes the private firms' deviation temptations and punishment payoffs in a non-linear fashion, causing the critical discount factor to fluctuate. This analytical dynamic is formally characterized in the following propositions:

Proposition 2.1

$$\frac{\partial \delta_q}{\partial \alpha}\begin{cases} \leq 0 & \textit{if } \alpha \leq 2-2\theta \\ \geq 0 & \textit{if } \alpha \geq 2-2\theta \end{cases}$$

Additionally, Proposition 2.2 demonstrates the non-monotonic relationship between δ_q and θ, whose behaviour oscillates between growth, decrease, and

resumption of growth in three sequential intervals limited by $(2-\alpha)/4$ and $(2-\alpha)/2$. This pattern of variation reflects the trade-off intrinsic to the degree of substitution in Cournot. At low levels of θ , the loss of differentiation outweighs strategic interdependence, increasing δ_q. In the intermediate range, the greater severity of punishment in symmetrical markets discourages deviation, reducing δ_q. As θ approaches 1, the near absence of differentiation makes defection optimal again, altering the direction of the system's derivative.

Given that the determining limits of these regions are strictly decreasing as a function of α, the interdependence between α and θ is evident. Indeed, the social welfare internalized by the public firm reconfigures the geometric structure of the relationship between product differentiation and the sustainability of collusion. This finding extends the classic debate on the ambivalent impact of horizontal differentiation on collusive stability Singh e Vives (1984), Deneckere (1983), Motta e Polo (2003), Belleflamme e Peitz (1945) to the scope of mixed oligopoly under quantity competition (Cournot).

Proposition 2.2

$$\frac{\partial \delta_q}{\partial \theta} \begin{cases} \geq 0 & \textit{if } \theta \in \left[0, \frac{2-\alpha}{4}\right] \cup \left[\frac{2-\alpha}{2}, 1\right] \\ \leq 0 & \textit{if } \theta \in \left[\frac{2-\alpha}{4}, \frac{2-\alpha}{2}\right] \end{cases}$$

The left panel of Figure 2 isolates a symmetric, concave trajectory for the critical discount factor (δ_q), illustrating how the instability of collusion reaches its peak at intermediate levels of product substitution (θ) as established by Proposition 2.1. Expanding on this, the right panel reveals that the curves for different ownership tiers intersect and invert their vertical ordering, a crossing behaviour that embodies the sign-switching property of $\partial \delta_q / \partial \theta$ and proves that privatization lacks a linear effect. Specifically, on the left side of the right panel where products are highly differentiated (low θ), greater public participation ($\alpha \to 1$) shifts the curves downward. Because quantities act as strategic substitutes, the public firm responds by expanding output aggressively to protect consumers, which severely depresses market prices during the non-cooperative punishment phase. This localized price depression reduces the private firms' punishment payoffs so drastically that it diminishes their relative losses from breaking the agreement, thereby compressing the payoffs, lowering the critical

threshold δ_q, and makes private collusion structurally easier to sustain than in a purely private market.

The results in Figure 2 reveal that the weight assigned to the public firm (α) and the degree of product differentiation (θ) affect the sustainability of collusion between private firms in a non-linear and interdependent way. Proposition 1 shows that the effect of α on δ_q changes sign at the threshold $\alpha = 2 - 2\theta$; below it, increases in social weight reduce the critical discount factor, since increasing the quantity of the public firm reduces the payoff of deviation more than the collusive payoff, facilitating tacit cooperation between private firms. Above the threshold, this effect changes, and the increase in α also erodes the value of the punishment, increasing δ_q and making it more difficult to sustain the agreement. This evidence suggests that the social function of the public firm acts as a stimulus to collusion to a lesser degree, but acts in a disciplinary way if it is sufficiently intense. A dual role that converges with discussions in the literature regarding collusive behaviour in mixed oligopoly structures (COLOMBO, 2013; MATSUMURA, 1998; SILVA; PINHO, 2018).

The comparison with the Bertrand model presented earlier reinforces this interpretation. Under price competition, greater product substitutability intensifies the demand response to relative price changes, thereby affecting the gain from a unilateral price deviation. Under Cournot competition, firms compete through quantities, so the effect of product substitutability operates through equilibrium quantities, strategic responses, and the resulting market price. The difference between the two models therefore concerns not only the magnitude of the critical discount factor, but also the strategic variable through which firms compete and the mechanism through which product substitutability affects collusive sustainability.

In this sense, the results are related to the classic argument of Singh e Vives (1984), according to which the choice between price and quantity competition affects equilibrium outcomes in markets with differentiated products. In the present model, this distinction acquires an additional dimension because the market includes a semi-public firm and the analysis concerns not only the static equilibrium, but also the conditions under which collusive behavior can be sustained in a repeated game. The contribution of the analysis lies in showing that, under Cournot quantity competition, the interaction between horizontal product differentiation and the semi-public firm's

social-welfare orientation affects the incentive-compatibility conditions of private firms differently from the Bertrand model.

3 FINAL CONSIDERATIONS

This dissertation aimed to evaluate the impact of the presence of a public firm on the sustainability of collusion between private firms in a mixed oligopoly with horizontal product differentiation, considering the Bertrand (price) and Cournot (quantity) competition regimes. Using a theoretical-analytical model of repeated games, in which two private firms maximize profits and a third, semi-public firm weighs profit and consumer surplus, the analysis examined how the degree of public participation (α) and the degree of product substitution (θ) alter the critical discount factor necessary to sustain a collusive agreement between the two private firms, supported by a grimm trigger strategy.

The results show that the presence of the public firm does not produce a uniform effect on the sustainability of collusion. In the Bertrand model, the relationship between the critical discount factor and the degree of product substitution exhibits non-monotonic behavior. When products are strongly differentiated ($\theta \to 0$) , the critical discount factor converges to 0.5; As substitutability increases, the need for patience to sustain the growing collusion grows, reaching its highest point at high levels of homogeneity and subsequently showing an inflection point when products approach perfect substitutability. Furthermore, the increase in public participation largely shifts the critical discount factor to higher levels in the analyzed domain, indicating greater difficulty in maintaining the collusive agreement. This result stems from the presence of the public firm oriented towards social welfare, whose pricing policy reduces the margins obtained by private firms and makes the decision to deviate relatively more attractive.

In the Cournot model, the results reveal a similarly non-linear relationship, although marked by distinct dynamics. The critical discount factor varies depending on the degree of product substitution, characterized by periods of growth, reduction, and subsequent growth within different theta intervals. Furthermore, the effect of public participation on the sustainability of the collusion depends jointly on the alpha and theta parameters. This effect can vary according to the relationship between these parameters: when public participation is below the threshold determined by this relationship, an increase can reduce the critical discount factor and, consequently, favor the sustainability of the collusion; when it is above this threshold, an increase in public

participation raises the critical discount factor, making the agreement more difficult to sustain. Therefore, in the Cournot regime, the public firm plays a dual role, being able, under certain structures, to contribute to the stability of the collusion and, under others, to exert a disciplining effect on private firms.

Regarding the degree of product differentiation, the results obtained in this dissertation show that its effect on the sustainability of collusion is not linear and depends on the competitive regime and the participation of the public firm. This result differs from that presented by Colombo (2016), according to whom a higher degree of product differentiation favors the sustainability of collusion, both by increasing the profitability of the agreement and by reducing the critical discount factor. In the present study, however, the relationship between differentiation and the sustainability of collusion is shown to be dependent on the competitive structure and the participation of the public firm, exhibiting non-linear behavior.

Taken together, these results lead to the conclusion that the relationship between ownership structure, product differentiation, and the sustainability of collusion is not linear and cannot be determined solely by the presence or absence of a public firm. The comparison between Bertrand and Cournot shows that the strategic variables, price and quantity, modify the intertemporal incentives of private firms and, consequently, the way in which public participation affects the critical discount factor. Thus, the answer to the central research question is conditioned by the competitive regime and the interaction between the degree of public participation and the sustainability of products. The public firm modifies the incentives for collusion, but does not, by itself, eliminate the possibility of coordination between private firms.

From the perspective of implications for competition policy and economic regulation, the results indicate that the evaluation of mixed markets should consider jointly the ownership structure, the degree of product differentiation, and the competition regime. The mere presence of a public firm is not sufficient to ensure the elimination of collusive behavior, since, particularly in the Cournot regime, certain combinations of public participation and differentiation can reduce the critical discount factor and make coordination between private firms relatively more sustainable. On the other hand, in other structures, the public firm's social welfare-oriented actions raise this factor and make maintaining the agreement more difficult.

As a perspective for future investigations, the results obtained indicate the relevance of deepening the analysis of the factors that condition the sustainability of collusion in mixed oligopolies, considering that the dissertation identifies limitations in the literature regarding the systematic comparison between price and quantity competition and regarding the interaction between public ownership and product differentiation. Further studies could, in this sense, expand the theoretical framework developed in this research and investigate other strategic structures capable of deepening the understanding of the relationship between public participation, competitive structure, and the stability of collusive agreements.

In summary, this dissertation demonstrates that the stability of collusion in mixed oligopolies results from the interaction between private incentives, the objectives of the public firm, the degree of substitutability of products, and the competitive mechanism. The results obtained from the Bertrand and Cournot models show that there is no single effect of public participation on collusion: its effects are dependent on the strategic conditions of the market. By highlighting this non-linear and interdependent nature, the study answers the research problem and contributes to a more precise understanding of the conditions under which the coexistence of public and private firms can favour or hinder the stability of collusive agreements.

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